

First to Market is not Everything: an Analysis of Preferential Attachment with Fitness

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Abstract

The design of algorithms on complex networks, such as routing, ranking or recommendation algorithms, requires a detailed understanding of the growth characteristics of the networks of interest, such as the Internet, the web graph, social networks or online communities. To this end, preferential attachment, in which the popularity (or relevance) of a node is determined by its degree, is a well-known and appealing random graph model, whose predictions are in accordance with experiments on the web graph and several social networks. However, its central assumption, that the popularity of the nodes depends only on their degree, is not a realistic one, since every node has potentially some intrinsic quality which can differentiate its attractiveness from other nodes with similar degrees.

In this paper, we provide a rigorous analysis of *preferential attachment with fitness*, suggested by Bianconi and Barabási and studied by Motwani and Xu, in which the degree of a vertex is scaled by its quality to determine its attractiveness. Including quality considerations in the classical preferential attachment model provides a much more realistic description of many complex networks, such as the web graph, and allows to observe a much richer behavior in the growth dynamics of these networks. Specifically, depending on the shape of the distribution from which the qualities of the vertices are drawn, we observe three distinct phases, namely a *first-mover-advantage* phase, a *fit-get-richer* phase and an *innovation-pays-off* phase. We precisely characterize the properties of the quality distribution that result in each of these phases and we compute the exact growth dynamics for each phase. The dynamics provide rich information about the quality of the vertices, which can be very useful in many practical contexts, including ranking algorithms for the web, recommendation algorithms, as well as the study of social networks. Furthermore, the mathematical techniques we introduce to establish these dynamics could be applicable to a wide variety of problems.

Keywords: random graphs, preferential attachment, Pólya urn processes, Bose-Einstein condensation

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1 Introduction

In recent years, there has been a convergence of ideas coming from computer science, social sciences and economic sciences as researchers in these fields attempt to model and analyze the characteristics and dynamics of large complex networks, such as the web graph, social networks and recommendation networks. From the computational perspective, it has been recognized that the successful design of algorithms performed on such networks, including routing, ranking and recommendation algorithms, must take into account the social dynamics as well as the technical properties and economic incentives that govern network growth [22, 23, 15].

Random Graph Models. An appealing way to model the growth dynamics of these networks is via random graph models. The well-studied Erdős-Rényi model is not an appropriate description of these networks, because it is a static rather than dynamic model, and more importantly, because sparse graphs drawn from the Erdős-Rényi model have Poisson degree distributions rather than the scale-free (power-law) distributions observed in a variety of social phenomena [26], and verified by experiments on the World Wide Web [2, 12, 16]—the latter seen as a massive graph with web pages being its vertices and directed edges between vertices corresponding to hyperlinks from one page to another.

Several models have been suggested which result in scale-free distributions, probably the first being due to Yule [25] and Simon [24]. In the context of scientific citations power law distributions were observed by Lotka [19], and Gilbert [13] specifies a probabilistic model supporting Lotka’s law. Kleinberg et al. [16] and Kumar et al. [18] suggest and study the *copy model* which captures the power law distribution and other connectivity properties of the World Wide Web, while other models include works from Broder et al. [8], Cooper and Frieze [9], Drinea et al. [11], Krapivsky and Redner [17].

Preferential Attachment Models. One of the most natural and attractive models for network growth is the *preferential attachment model*, suggested by Barabási and Albert [2] to model the web graph, and originally proposed as the *cumulative advantage* model by Derek de Solla Price in 1965 [10]. See e.g. [7, 6] for a rigorous treatment. Roughly speaking, as time evolves, new vertices join the network by adding several links to the vertices already present in the network in a probabilistic fashion. The probability of linking to an existing vertex is an increasing function, usually polynomial, in its degree, which captures the intuitive fact that higher degree of a vertex reflects higher relevance or popularity.

This model by itself has been rather successful in predicting the graph structure of the web [2], at least as an undirected graph. Nevertheless, there is an unsatisfactory assumption underlying the model. The popularity of a vertex depends only on its degree. As a result, the prediction of the model is the so-called *first-mover-advantage* phenomenon in which earlier vertices tend to have significantly higher degrees than later ones, making it hard for a vertex which enters late to compete with the already established *hubs* of the network. Moreover, the model is completely symmetric with respect to vertices which enter at similar times, since there is no modeling of how the intrinsic quality of every vertex affects its growth in the network. How is the quality of vertices reflected in the network structure and its dynamics? How can one extract such information?

To answer this type of questions we analyze a variant of the preferential attachment model which explicitly models the intrinsic quality of the vertices. This model, introduced in the context of the web by Bianconi and Barabási [4], is usually called *preferential attachment with fitness*. In this model, when a new vertex is created, it gets assigned a quality parameter, henceforth called *fitness*, drawn from a given distribution, which scales its degree to determine its attractiveness in the evolution of the network. The resulting model provides a much more accurate description of many real-world networks [4], but it is also more difficult to analyze rigorously; see Bianconi and Barabási [4] for

heuristic arguments and Motwani and Xu [21] for more precise—but nevertheless heuristic in several aspects—arguments.

Our Results. We provide the first—to our knowledge—rigorous analysis of preferential attachment with fitness. We show that, depending on the properties of the distribution from which the fitnesses are drawn, henceforth called the *fitness distribution*, there is a much richer behavior than what is predicted by the classical preferential attachment model. We precisely characterize the possible evolutions of a complex network and we specify the properties of the fitness distribution resulting in each of them. More precisely, we show that, depending on the fitness distribution, an evolving network can undergo one of the following behaviors, or *phases*:

- the *first-mover-advantage* phase, which results from flat fitness distributions and corresponds to the power-law behavior predicted by the classical preferential attachment model;
- the *fit-get-richer* phase, in which vertices of higher fitness grow faster than those of smaller fitness; the behavior here is a power-law within each fitness value, but the tail exponent decreases as the fitness increases;
- the *innovation-pays-off* phase, in which roughly speaking the competition for links results in a constant fraction of the links continuously shifting to ever larger fitness values; this fraction of links that “escapes to infinity” is independent of the network size and is determined by the fitness distribution; such behavior is not observed in the fit-get-richer phase.

Our analysis is applicable to both discrete and continuous fitness distributions, as well as bounded or unbounded ones, and we provide precise criteria for the fitness distribution that specify which of the above phases will arise. In fact, we discover some property of the fitness distribution which exhibits a sharp phase transition separating the latter evolution scenarios. Our results are in accordance with the predictions of Bianconi and Barabási [4] derived by mapping the evolving network to a Bose gas in the thermodynamic limit. In this terminology, the innovation-pays-off phase corresponds to the phenomenon of *Bose-Einstein condensation*, whereby a constant fraction of the particles condensate on the lowest energy level, corresponding in the network context to the supremum of the fitness values.

A by-product of our technique is a precise characterization of the *vertex dynamics* under preferential attachment with fitness. More specifically, if a vertex v has fitness f , then our analysis implies that its degree $d_v(t)$ at time t scales as

$$d_v(t) \sim t^{cf}, \quad (1)$$

where c is a global constant determined by the fitness distribution. Hence, the logarithm of the degree of the vertices directly reflects their quality. This could suggest new directions in the design of ranking or recommendation algorithms.

Proof Techniques. The standard approach to analyze preferential attachment models is to derive recursions (or differential equations), typically, of the expected number of nodes of a given degree. See e.g. [20]. This type of technique relies crucially on the fact that the number of nodes at any time in the graph is deterministic—a quantity that arises as the denominator in the recursion. However, in our case, the relevant quantity is the number of nodes *weighted by their fitness* which, unfortunately, is a *random* variable. This turns out to complicate significantly the analysis.

To obtain our results, we rely instead on a very different approach, one based on the theory of Pólya urn models. In Pólya’s classical urn scheme, an urn contains balls of two colors. At each time step, a ball is drawn randomly from the urn and returned along with an extra ball of the same color. This is clearly reminiscent of a preferential attachment scheme and the connection between the two models has previously been exploited, e.g. in [3]. Here we use a generalized version of Pólya’s

scheme (see e.g. [14]): 1) we consider an arbitrary, but finite number of colors; 2) each ball is picked proportionally to a weight, or “activity parameter”, associated to its color; and 3) at each time step, the ball picked is returned along with a random number of balls of each color, where the distribution of this “random update vector” depends on the color of the ball drawn.

We analyze the limiting behavior of the preferential attachment scheme with fitness by coupling the growth process with specially crafted generalized Pólya urn models where the colors represent connectivity properties of the evolving network, e.g. the cumulative degree of all vertices of a given fitness. When the fitness distribution is concentrated on a finite number of atoms, the correspondence is somewhat straightforward, although our coupling appears to be novel and it allows to derive nontrivial generalizations of classic results very easily. More importantly, we consider in fact general fitness distributions, including continuous distributions, which in principle require an infinite number of colors in the Pólya urn model. Little is known about the behavior of generalized Pólya urns beyond the finite case, and we resort to various novel truncation techniques to map the dynamics of our network to a finite urn process. We expect that our techniques should be useful in a much more general context to the analysis of previously unapproachable complex network growth models, which now may be analyzed using infinite Pólya urn models with techniques analogous to those developed here.

1.1 Definitions and Main Result

The Model. The generalized preferential attachment model of Bianconi and Barabási which we analyze here is a random graph model defined as follows.

Definition 1 (Preferential Attachment Scheme with Fitness) *Let $\mathcal{F} \subseteq \mathbb{R}_+$ be a set of fitnesses and \mathcal{Q} a distribution over fitnesses such that $\int_{\mathcal{F}} d\mathcal{Q}(f) = 1$. The preferential attachment process with fitness begins with one vertex of fitness $f \in \mathcal{F}$ drawn according to \mathcal{Q} and a self-loop on that vertex. Then, at every time step t , a new vertex is added to the graph, which has fitness picked independently according to \mathcal{Q} and is attached to an old vertex v with probability proportional to $f_v \cdot d_{v,t-1}$, where f_v is the fitness of vertex v and $d_{v,t-1}$ its degree at step $t-1$. We denote by $G_n = (V_n, E_n)$ the graph at time n . We sometimes refer to this process as the $(\mathcal{F}, \mathcal{Q})$ -chain.*

It turns out that the case of unbounded fitnesses is rather uninteresting (see Appendix C.4) and hereon we assume that $\sup\{f : f \in \mathcal{F}\} = h$ for some $h < +\infty$. Furthermore, we consider three main cases for \mathcal{F} : either \mathcal{F} is discrete—finite or countable—with \mathcal{Q} strictly positive on \mathcal{F} , or \mathcal{F} is the interval $[0, h]$ and \mathcal{Q} admits a strictly positive continuous density on $(0, h)$. We say that $(\mathcal{F}, \mathcal{Q})$ is *regular* in such cases. Our results extend to more general fitness distributions but we restrict ourselves to the regular case here. Also, the process above constructs only undirected trees. However, our techniques can be easily extended to directed scale-free graphs as defined in [5]. We omit the details.

Main Result. Our basic result concerns the distribution of links across fitnesses as $n \rightarrow +\infty$. Let $[a, b] \subseteq [0, h]$ with $a \leq b$ and denote by $M_{n,[a,b]}$ the number of edge endpoints with fitness in $[a, b]$ in G_n . Let λ_0 be the (unique) solution in $[h, +\infty)$ of

$$\mathcal{I}(\lambda_0) \equiv \int_{\mathcal{F}} \frac{f}{\lambda_0 - f} d\mathcal{Q}(f) = 1, \quad (2)$$

if it exists and let $\lambda_0 = h$ otherwise. Our main result is the following.

Theorem 1 (Basic Result) *Assume $(\mathcal{F}, \mathcal{Q})$ is regular. Then, for all $[a, b] \subseteq [0, h]$ with $a \leq b$, we have*

$$\frac{M_{n,[a,b]}}{n} \rightarrow \lambda_0 \int_{\mathcal{F} \cap [a,b]} \frac{1}{\lambda_0 - f} d\mathcal{Q}(f) \equiv \nu_{[a,b]}, \quad \frac{M_{n,[a,h]}}{n} \rightarrow 2 - \lambda_0 \int_{\mathcal{F} \cap [0,a]} \frac{1}{\lambda_0 - f} d\mathcal{Q}(f) \equiv 2 - \nu_{[0,a]},$$

almost surely as $n \rightarrow +\infty$.

A surprising behavior arises when (2) has no solution in $[h, +\infty)$, or equivalently when $\mathcal{I}(h) < 1$. Indeed, in such a case, it is easy to check that $\nu_{[0, h-\varepsilon]} \leq 1 + \mathcal{I}(h) < 2$ for all $\varepsilon > 0$ even though we expect $\lim_{\varepsilon \rightarrow 0} \nu_{[0, h-\varepsilon]} = 2$ since for all n , $n^{-1}M_{n, [0, h]} = 2$ (i.e. each edge has two endpoints). In other words, it appears that a constant fraction of edges is “missing” in the limit. The missing fraction actually “escapes to h ” which leads to what we call the innovation-pays-off phase as described above. To get a better intuition for the existence of a solution in (2), consider the example $\mathcal{Q} \sim \text{Beta}(\alpha, \beta)$. In Example 4 of Appendix C.3, we show there is a solution if and only if $\beta \leq \alpha + 1$. For a fixed α , a large β indicates a “fast decay” to 0 at 1 while a small β leads to a “fatter tail” around 1. A solution to (2) exists in the latter case, e.g. in the uniform case. In other words, the innovation-pays-off regime requires a more “rarefied” high fitness population.

Dynamics of the Innovation-Pays-Off Phase. In order to understand (informally) the dynamics of the innovation-pays-off phase, fix a time t^* and let f^* be the largest fitness among vertices present in the network at time t^* . Note that

- at time t^* , the cumulative fraction of the links shared by vertices of fitness up to f^* is 2, since every edge is accounted for twice;
- now, consider the network in the limit $t = +\infty$; by Theorem 1 and the discussion above, the fraction of links shared among vertices of fitness up to f^* is at most $1 + \mathcal{I}(h)$; therefore at least a fraction $1 - \mathcal{I}(h)$ of links is shared among vertices of fitness larger than f^* , vertices which, by definition, were not present at time t^* .

This is the “signature” of the innovation-pays-off phase: a constant fraction of the links changes hands toward higher and higher fitness values.

Power Laws and Vertex Dynamics. In fact, we can prove more than Theorem 1. As stated below in Theorems 3 and 4 and their counterparts in the continuous case, we exhibit power laws for the degree distributions on the nodes of a given fitness and we get a tail exponent of $\lambda_0 f^{-1}$ where f is the given fitness. See Section 4. Also, as discussed above, we can prove vertex dynamics of the form (1). Such result is proved by considering a continuous-time embedding of the process as in [14]. Details are omitted. The constant c in (1) is in fact λ_0^{-1} .

Proof Sketch. As we mentioned before, the basic idea of the proof of Theorem 1 (as well as of the power law results in Theorems 3 and 4 below) is to couple the preferential attachment process with Pólya urn models. The first step is the analysis of the case \mathcal{F} finite. There we proceed by truncating large degrees and associating a color of a specially designed Pólya process to each pair (degree, fitness). The limit theory of Pólya processes then reduces the problem to an eigenvector computation of an appropriately defined matrix (see Section 2). This computation appears to be tricky but turns out to be manageable, as described in Appendix A.

The countable and continuous cases are significantly more challenging since Pólya urns with infinite—whether countable or uncountable—colors are poorly understood. Instead, we use further truncation and approximation techniques to couple the infinite cases with finite cases. In Section 4, we illustrate this idea on the somewhat easier special case of $\mathcal{F} = \{f_j\}_{j=1}^{+\infty}$ increasing. There we need two finite Pólya models—a lower bound and an upper bound—which are obtained by truncating \mathcal{F} and mapping the remaining fitness values to either 0 or h . The general discrete case as well as the continuous case require a much more sophisticated approach which is detailed in Appendix C.

Organization of the Paper. We start with a brief overview of generalized Pólya urn models in Section 2 followed by our treatment of preferential attachment for finite fitness distributions in

Section 3. The main steps of the general proof are illustrated in Section 4 in the special case where $\mathcal{F} = \{f_j\}_{j \geq 1}$ is countable and increasing. Most proofs are relegated to the appendix. Most notably, for lack of space the particularly interesting analysis of the continuous case is *completely* relegated to Appendix C.

Notation. We denote by e_i the unit vector along the i -th axis (usually the dimension is clear). The notation $\mathbf{1}_S$ denotes the indicator of the event S .

2 Generalized Pólya Urns

Our results are obtained through an appropriate mapping of the preferential fitness process to a finite generalized Pólya urn scheme. We introduce here the basic limit theory of generalized Pólya urn models keeping our notation consistent with the presentation of Janson [14], with the exception of our matrix A which is the transpose of Janson's, in accordance with common practice in the Pólya urn literature.

Definition of the Pólya Urn Process. We have $q < +\infty$ bins (corresponding to the colors in the original Pólya model described in the Introduction). Each bin $i \leq q$ is assigned a fixed activity a_i , $0 \leq a_i < +\infty$. For $n \geq 0$, let

$$X_n = (X_{n,1}, \dots, X_{n,q}),$$

where $X_{n,i}$ is the number of balls in bin i at time n . The initial load is given by X_0 , which may be random or deterministic. Each bin, say i , also has a random vector $\xi_i = (\xi_{i,1}, \dots, \xi_{i,q})$ with integer coordinates. The process is defined as follows. At time n , we pick one bin. Bin i is chosen with probability proportional to $a_i X_{n-1,i}$. If bin i is picked, we draw an independent copy $\xi_i^{(n)}$ of ξ_i and update $\{X_n\}_{n \geq 0}$ according to

$$X_n = X_{n-1} + \xi_i^{(n)}.$$

Basic Pólya Urn Result. The limiting behavior of the Pólya Urn process described above can be characterized in terms of the $q \times q$ matrix A with entries

$$A_{i,j} = a_i \mathbb{E}[\xi_{i,j}],$$

assuming conditions (A1)-(A6) in [14] are satisfied. In fact, we will only need to use the more general assumption described in Remark 4.2 of [14]. Roughly speaking, we require that:

- The urn process is well-defined (see the definition of *tenable* in Remark 4.2 of [14]). Essentially, we require that the number of balls remains nonnegative at all times with probability 1.
- The matrix A satisfies a slight generalization of irreducibility and the initial load is positive on a “dominating type.” This generalization allows for dummy bins that “count certain events.” (See Section 3 “Limits for urns” of [14].)
- The vectors ξ_i have finite second moments. In our application, the ξ_i 's will actually be bounded.

We refer the reader to [14] for more details. Under these conditions, it is not hard to see that A has a unique largest positive eigenvalue λ_1 with corresponding positive left eigenvector v_1 and right eigenvector u_1 (apply the Perron-Frobenius theorem to $A + \alpha I$ for an appropriate α). We choose u_1, v_1 to satisfy $a \cdot v_1 = 1$ and $u_1 \cdot v_1 = 1$ where a is the vector of activities. The following theorem characterizes the vector X_n .

Theorem 2 (Limit of Finite Urns [1]; Theorem 3.21 in [14]) *Assume conditions (A1)-(A6) of [14] are satisfied. Conditioned on essential non-extinction (see [14]) we have*

$$\frac{X_n}{n} \rightarrow \lambda_1 v_1,$$

almost surely as $n \rightarrow +\infty$.

In our applications of Theorem 2, it will be easy to establish that “essential extinction” is not possible.

3 Preferential Attachment: Finite Distributions

In this section, we treat the case $\mathcal{F} = \{f_j\}_{j \in J}$ where J is finite—which we sometimes refer to as the *finite-type case*. This will form the basic step in the analysis of the countable and continuous cases. Without loss of generality, we take $\{f_j\}_{j \in J}$ increasing. We analyze separately the distribution of degrees within each fitness value (Section 3.1) and the distribution of links across fitness values (Section 3.2). We then combine the two results in Section 3.3. Note that, as we describe below, only the first-mover-advantage and fit-get-richer behaviors arise in the finite-type case.

3.1 Flat Fitness Distributions: First-Mover-Advantage

Suppose first that $J = 1$. This is the standard preferential attachment model, which is well understood (see e.g. [20] and references therein). We rederive the degree distribution by first mapping to a Pólya urn process and then applying Theorem 2. The mapping is illustrative of our technique. Let $L_{n,k}$ be the number of vertices of degree k at time n ; set $\mu_1 = \frac{2}{3}$ and, for $k \geq 2$,

$$\mu_k = \frac{2}{3} \prod_{l=2}^k \frac{l-1}{l+2} = \frac{4}{k(k+1)(k+2)} \sim k^{-3}.$$

In particular, $\{\mu_k\}_{k \geq 1}$ is a *power law with tail exponent 2*.

Proposition 1 (1-Fitness Case; see e.g. [20]) *For all $k \geq 1$,*

$$\frac{L_{n,k}}{n} \rightarrow \mu_k$$

almost surely as $n \rightarrow +\infty$.

Proof: Fix $k \geq 1$ and consider the following urn process with $k+1$ urns of equal activities $a_i = 1$, for all $1 \leq i \leq k+1$. We will design the process in such a way that the number of balls in urn i at time n represents the number of edges in the graph which are adjacent to vertices of degree i —counting twice edges with both endpoints at vertices of degree i . Except for the $(k+1)$ -st urn, where the number of balls will represent the number of edges adjacent to vertices of degree $\geq k+1$.

Let $X_0 = (0, 2, 0, \dots, 0)$ reflecting the fact that initially there is a single vertex with a self loop (degree 2). For $2 \leq i \leq k$, let the update vector ξ_i be deterministic with

$$\xi_{i,j} = \begin{cases} 1, & j = 1 \\ -i, & j = i \\ i+1, & j = i+1 \\ 0, & \text{o.w.} \end{cases}$$

reflecting the fact that, if the new vertex being added to the graph links to an old vertex of degree i , then the degree of that vertex becomes $i+1$, therefore the edges adjacent to that vertex must be

accounted for in the urn $i + 1$ instead of the urn i . Finally, for urns $i = 1$ and $i = k + 1$, the following update vectors respect the boundary conditions

$$\xi_{1,j} = \begin{cases} 2, & j = 2 \\ 0, & \text{o.w.} \end{cases} \quad \text{and} \quad \xi_{k+1,j} = \begin{cases} 1, & j = 1 \\ 1, & j = k + 1 \\ 0, & \text{o.w.} \end{cases}$$

It is not hard to see that the urn process described above can be coupled with the preferential attachment process so that with probability 1 the following relations are satisfied, for all $n \geq 0$,

$$\begin{cases} X_{n,\ell} = \ell L_{n,\ell}, & \text{for } 1 \leq \ell \leq k \\ X_{n,k+1} = \sum_{\ell \geq k+1} \ell L_{n,\ell} \end{cases}$$

The proof is concluded by computing matrix A , its largest eigenvalue λ_1 and the corresponding left eigenvector v_1 (see Appendix A). One can check that Conditions (A1)-(A6) of [14] are satisfied. ■

3.2 Competition for Links across Fitness Values

We now consider the case $J = |\mathcal{F}| > 1$ finite. We aim to compute the limiting behavior of the random variables $M_{n,j}$, $1 \leq j \leq J$, corresponding to the number of edges with an endpoint of fitness f_j at time n —counting twice edges with two endpoints of fitness f_j , i.e. the total degree of vertices of fitness f_j . Let $\lambda_0 > 0$ be the largest solution to the equation

$$\sum_{j=1}^J \frac{f_j q_j}{\lambda_0 - f_j} = 1, \quad (3)$$

where, by monotonicity, $\lambda_0 \in (\max_j \{f_j\}, +\infty)$. Also, for $1 \leq j \leq J$, set

$$\nu_j = \lambda_0 \frac{q_j}{\lambda_0 - f_j}, \quad (4)$$

and verify that

$$\sum_{j=1}^J \nu_j = \sum_{j=1}^J (\lambda_0 - f_j) \frac{q_j}{\lambda_0 - f_j} + \sum_{j=1}^J f_j \frac{q_j}{\lambda_0 - f_j} = 2.$$

We characterize the distribution of links across fitness values in terms of the ν_j 's.

Proposition 2 (Fitness Alone) *For all $1 \leq j \leq J$,*

$$\frac{M_{n,j}}{n} \rightarrow \nu_j,$$

almost surely as $n \rightarrow +\infty$.

Proof: We define the following urn process with J urns in which urn $i \leq J$ has activity $a_i = f_i$. The urn process will be designed so that the number of balls in urn i corresponds to the number of edges with an endpoint of fitness f_i . For $1 \leq i \leq J$, the update vector ξ_i is given by $\xi_i = e_i + \Delta_i$, where $\Delta_i = e_j$ with probability q_j , for all $1 \leq j \leq J$. In the context of the preferential attachment process, this reflects the fact that, if the new vertex links to a bin of fitness f_i , then the number of edges with an endpoint of fitness f_i increases by one, hence the term e_i ; moreover, the new vertex picks a random fitness according to \mathcal{Q} , hence the term Δ_i . It is easy to couple the defined urn process with the preferential attachment one so that, with probability 1, $X_{n,j} = M_{n,j}$, for all $1 \leq j \leq J$ and all $n \geq 0$, provided $X_0 = 2e_i$ with probability q_i . The proof is concluded by computing matrix A , its largest eigenvalue λ_1 and the corresponding left eigenvector v_1 (see Appendix A). ■

3.3 Finite Distributions: Fit-Get-Richer

In this section, we derive the degree distribution of preferential attachment with fitness under finite fitness distributions. For all $1 \leq j \leq J$ and $k \geq 1$, denote by $N_{n,(j,k)}$ the number of vertices of fitness f_j and degree k at time n . Define λ_0 and $\{\nu_j\}_{j=1}^J$ as in Section 3.2. Moreover, for all $1 \leq j \leq J$ and $k \geq 1$, set $\eta_{(j,k)}$ as follows

$$\eta_{(j,k)} = \nu_j \cdot \frac{1}{k} \prod_{\ell=2}^k \frac{\ell}{\ell + \lambda_0 f_j^{-1}}. \quad (5)$$

In particular,

$$\frac{\eta_{(j,k+1)}}{\eta_{(j,k)}} = \frac{k}{k+1} \frac{k+1}{k+1 + \lambda_0 f_j^{-1}} = 1 - \frac{1 + \lambda_0 f_j^{-1}}{k} (1 + o(1)),$$

as k gets large. Thus, for fixed j , $\{\eta_{(j,k)}\}_{k \geq 1}$ has tail exponent $\lambda_0 f_j^{-1}$.

Proposition 3 (Finite Fitness Distributions: Fit-Get-Richer) *For all $1 \leq j \leq J$ and $k \geq 1$, we have*

$$\frac{N_{n,(j,k)}}{n} \rightarrow \eta_{(j,k)},$$

almost surely as $n \rightarrow +\infty$.

Observe that the tail exponent is a decreasing function of the fitness. Hence, the tail of the distribution gets fatter as the fitness increases. This is the “signature” of the fit-get-richer phase. The proof of Proposition 3 is postponed to the appendix. It follows from a combination of the couplings in Propositions 1 and 2, by defining a Pólya urn process with a bin for every pair of fitness and degree. Once again, the degree is truncated at a maximum value and an extra bin accounts for all degrees above.

4 Preferential Attachment: Countable Distributions

If $J = +\infty$, which we sometimes call the infinite-type case, the coupling described in the previous section cannot be used directly, since it would then require an infinite number of urns (for the fitnesses alone) and Theorem 2 is not known to hold generally in the infinite case. Nevertheless, we obtain similar results by coupling our process this time with *two* finite-type preferential attachment processes which provide lower and upper bounds on the degree distribution of our process. The coupling is presented in Section 4.1. Using this coupling and Proposition 3, we exhibit the following evolution scenarios for the preferential attachment process with countable fitness distribution:

- the *fit-get-richer scenario*, taking place when $\sum_{j=1}^{+\infty} \frac{f_j q_j}{h - f_j} \geq 1$,
- the *innovation-pays-off scenario*, taking place when $\sum_{j=1}^{+\infty} \frac{f_j q_j}{h - f_j} < 1$,

where $h = \sup_{j \geq 1} \{f_j\}$.

For convenience, we treat only the case $\{f_j\}_{j \geq 1}$ increasing. The general case—which is omitted from this extended abstract—follows from an analysis similar to that for continuous fitness distributions in Appendix C.

4.1 Coupling

Denoting by h the supremum of $\{f_j\}_{j \geq 1}$, let us assume that $h < +\infty$; the case $h = +\infty$ is treated in Section B.4 of the appendix. Setting I to be a positive integer, the *upper I -truncation* of \mathcal{F} , denoted

$\overline{\mathcal{F}} = \{\overline{f}_j\}_{j \geq 1}$, and the *lower I -truncation* of \mathcal{F} , denoted $\underline{\mathcal{F}} = \{\underline{f}_j\}_{j \geq 1}$, are defined by

$$\overline{f}_j = \begin{cases} f_j, & j \leq I \\ 0, & \text{o.w.} \end{cases} \quad \underline{f}_j = \begin{cases} f_j, & j \leq I \\ h, & \text{o.w.} \end{cases}$$

We shall couple the $(\mathcal{F}, \mathcal{Q})$ chain with the chains $(\overline{\mathcal{F}}, \mathcal{Q})$, $(\underline{\mathcal{F}}, \mathcal{Q})$ defined by the upper and lower truncations to provide upper and lower bounds respectively on the degrees of chain $(\mathcal{F}, \mathcal{Q})$ ¹. Roughly speaking, the chains can be coupled so that, at every step, the probability of choosing an old vertex of fitness value f_1 up to f_J is larger in the $(\overline{\mathcal{F}}, \mathcal{Q})$ than in the $(\mathcal{F}, \mathcal{Q})$ chain and larger in the $(\mathcal{F}, \mathcal{Q})$ than in the $(\underline{\mathcal{F}}, \mathcal{Q})$ chain. This property certainly holds in the beginning of the processes and then reproduces itself since it makes the cumulative degree of fitness levels f_1 up to f_J grow faster in the $(\overline{\mathcal{F}}, \mathcal{Q})$ than in the $(\mathcal{F}, \mathcal{Q})$ chain and faster in the $(\mathcal{F}, \mathcal{Q})$ than in the $(\underline{\mathcal{F}}, \mathcal{Q})$ chain. It is important to note however that the degree by itself is not sufficient to guarantee the domination of probabilities for the next step of the process; rather we couple the edges which get added at each step in such a way that the fitness values of the endpoints in chain $(\underline{\mathcal{F}}, \mathcal{Q})$ dominate the fitness values in $(\mathcal{F}, \mathcal{Q})$ and those dominate the fitness values in chain $(\overline{\mathcal{F}}, \mathcal{Q})$.

Fitness Alone. We first bound $M_{n,j}$, defined as in Section 3.2 to be the number of edges with an endpoint of fitness f_j (counting twice edges with two endpoints of fitness f_j). Fixing $1 < I < +\infty$, let $\overline{M}_{n,j}$ and $\underline{M}_{n,j}$ be the corresponding variables of the $(\overline{\mathcal{F}}, \mathcal{Q})$, $(\underline{\mathcal{F}}, \mathcal{Q})$ chains. It is clear that the latter are equivalent to finite type urn processes, so that Proposition 2 applies. Let $\overline{\nu}_j$ and $\underline{\nu}_j$ be the (almost sure) limits of $n^{-1}\overline{M}_{n,j}$ and $n^{-1}\underline{M}_{n,j}$. Then we have the following.

Lemma 1 (Coupling: Fitness Alone) *For all $1 \leq j \leq I$, it holds almost surely that*

$$\limsup_{n \rightarrow +\infty} \frac{M_{n,j}}{n} \leq \overline{\nu}_j, \quad \text{and} \quad \liminf_{n \rightarrow +\infty} \frac{M_{n,j}}{n} \geq \underline{\nu}_j.$$

Proof: Consider the $(\mathcal{F}, \mathcal{Q})$ -chain. At step $n \geq 1$, a vertex is picked with probability proportional to its degree scaled by its fitness. Let F_n be the fitness of the chosen vertex and denote by $\rho_{n-1,i}$ the probability that $F_n = f_i$ given the state of the chain after step $n-1$. After a vertex is picked, a new vertex is added with fitness chosen according to \mathcal{Q} . Let F'_n be the fitness of this new vertex. Denote by $\overline{F}_n, \overline{F}'_n, \overline{\rho}_n, \underline{F}_n, \underline{F}'_n, \underline{\rho}_n$ the corresponding variables for the chains $(\overline{\mathcal{F}}, \mathcal{Q})$ and $(\underline{\mathcal{F}}, \mathcal{Q})$ respectively. We define a coupling of the three chains so as to preserve the following conditions:

1. For all $n \geq 1$, $\overline{F}_n \leq F_n \leq \underline{F}_n$ and $\overline{F}'_n \leq F'_n \leq \underline{F}'_n$.
2. For all $n \geq 1$ and all $1 \leq i \leq I$, $\underline{M}_{n,i} \leq M_{n,i} \leq \overline{M}_{n,i}$.
3. For all $n \geq 1$ and all $1 \leq i \leq I$, $\underline{\rho}_{n,i} \leq \rho_{n,i} \leq \overline{\rho}_{n,i}$.

Note that 3. follows immediately from 1. and 2. We now justify why the conditions are satisfied for all $n \geq 0$. The initial configuration ($n = 0$) is constructed by picking an i according to \mathcal{Q} and choosing the corresponding fitness in all three chains. Therefore the conditions are satisfied at time 0 by the definition of $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$. Assuming that Conditions 1., 2., and 3. are satisfied at time $n-1$ we will show that they are true at time n . Indeed, since the fitness of the new vertex is picked according to \mathcal{Q} in all 3 chains it follows from the definition of $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$ that $\overline{F}'_n \leq F'_n \leq \underline{F}'_n$. Now let us consider the step of picking the old vertex. By 3., it follows that the choices made in the three chains can be coupled so as to satisfy Conditions 1. and 2. Indeed, proceed as follows:

- with probability $\sum_{i=1}^I \underline{\rho}_{n-1,i}$, pick the same fitness in all three chains according to $\{(\underline{\rho}_{n-1,i})\}_{i=1}^I$;

¹Strictly speaking, we think of \mathcal{Q} here as a distribution on the *indices* of the fitness sequences \mathcal{F} , $\underline{\mathcal{F}}$, and $\overline{\mathcal{F}}$ rather than on the fitnesses themselves.

- with probability $\sum_{i=1}^I (\rho_{n-1,i} - \underline{\rho}_{n-1,i})$, pick the same fitness in chains $(\mathcal{F}, \mathcal{Q})$ and $(\overline{\mathcal{F}}, \mathcal{Q})$ according to $\{(\rho_{n-1,i} - \underline{\rho}_{n-1,i})\}_{i=1}^I$ and some fitness h for $(\underline{\mathcal{F}}, \mathcal{Q})$;
- with probability $\sum_{i=1}^I (\overline{\rho}_{n-1,i} - \rho_{n-1,i})$, pick a fitness for the $(\overline{\mathcal{F}}, \mathcal{Q})$ -chain according to $\{(\overline{\rho}_{n-1,i} - \rho_{n-1,i})\}_{i=1}^I$, pick some fitness h for $(\underline{\mathcal{F}}, \mathcal{Q})$, and pick a fitness for $(\mathcal{F}, \mathcal{Q})$ according to $\{(f_j M_{n,j})\}_{j>I}$;
- note that there is no remaining probability mass since $\sum_{i=1}^I \overline{\rho}_{n-1,i} = 1$.

This concludes the proof. It should be clear that the described coupling is valid. ■

Full Analysis. Using our coupling idea we can also derive bounds on $N_{n,(j,k)}$, defined as in Section 3.3 to be the number of vertices of fitness f_j and degree k at time n in the $(\mathcal{F}, \mathcal{Q})$ -chain, in terms of the corresponding variables of the $(\overline{\mathcal{F}}, \mathcal{Q})$ -chain and $(\underline{\mathcal{F}}, \mathcal{Q})$ -chain. The coupling has a similar flavor and its details are postponed to Section B.1 of the appendix.

4.2 Fit-Get-Richer Phase

Let $h = \sup_{j \geq 1} f_j < +\infty$, the case $h = +\infty$ being treated in Section B.4. Unlike the finite-type case, when $J = +\infty$, we are not guaranteed that there exists a solution of

$$\sum_{j=1}^J \frac{f_j q_j}{\lambda - f_j} = 1, \quad (6)$$

with $\lambda > h$. Observe, however, that in our proof of Proposition 2 this was necessary for the existence of a (summable) Perron-Frobenius eigenvector (see the expression for v_1 in the proof of Proposition 2). We will actually show that the behavior of the process depends crucially on the existence of such a solution. In this section, we consider the case

$$\sum_{j=1}^J \frac{f_j q_j}{h - f_j} > 1. \quad (7)$$

We generalize Proposition 3 exhibiting a fit-get-richer behavior in this case. The following theorem summarizes our result.

Theorem 3 (Discrete Case: Fit-Get-Richer Phase) *Let $1 \leq J \leq +\infty$, $h = \sup_{j \geq 1} f_j < +\infty$.*

Assume
$$\sum_{j=1}^J \frac{f_j q_j}{h - f_j} > 1.$$

Then it holds that

1. *for all $1 \leq j < J + 1$, $\frac{M_{n,j}}{n} \rightarrow \nu_j$, almost surely as $n \rightarrow +\infty$,*
2. *for all $1 \leq j < J + 1$ and $k \geq 1$, $\frac{N_{n,(j,k)}}{n} \rightarrow \eta_{(j,k)}$, almost surely as $n \rightarrow +\infty$,*

where $\{\nu_j\}_j$ and $\{\eta_{(j,k)}\}_{j,k}$ are defined by Equations (4), (5).

4.3 Innovation-Pays-Off Phase

Assume that $h = \sup_{j \geq 1} f_j < +\infty$ and that

$$\mathcal{I}(h) \equiv \sum_{j=1}^J \frac{f_j q_j}{h - f_j} \leq 1. \quad (8)$$

It is easy to check that this is possible only if the fitness supremum h is not attained in \mathcal{F} (see also the discussion in Example 1 of the appendix). In particular, it must be that $J = +\infty$. Now set $\nu'_j = h \frac{q_j}{h - f_j}$, for $1 \leq j < +\infty$, and note in particular that

$$\sum_{j=1}^{+\infty} \nu'_j = \sum_{j=1}^{+\infty} (h - f_j) \frac{q_j}{h - f_j} + \sum_{j=1}^{+\infty} f_j \frac{q_j}{h - f_j} = 1 + \mathcal{I}(h) \leq 2, \quad (9)$$

with equality only if there is equality in (8)². Also, for all $1 \leq j < +\infty$ and $k \geq 1$, let $\eta'_{(j,k)}$ be defined as $\eta'_{(j,k)} = \frac{h q_j}{k(h - f_j)} \prod_{l=2}^k \frac{l}{l + h f_j^{-1}}$. In particular, $\frac{\eta'_{(j,k+1)}}{\eta'_{(j,k)}} = \frac{k}{k+1} \frac{k+1}{k+1 + h f_j^{-1}} = 1 - \frac{1 + h f_j^{-1}}{k} (1 + o(1))$, as k gets large. Hence, for fixed j , $\{\eta'_{(j,k)}\}_{k \geq 1}$ has tail exponent $h f_j^{-1}$.

Theorem 4 (Discrete Case: Innovation-Pays-Off Phase) *Let $h = \sup_{j \geq 1} f_j < +\infty$. Assume*

$$\sum_{j=1}^{+\infty} \frac{f_j q_j}{h - f_j} \leq 1. \quad (10)$$

Then it holds that

1. *For all $1 \leq j < +\infty$, $\frac{M_{n,j}}{n} \rightarrow \nu'_j$, almost surely as $n \rightarrow +\infty$.*
2. *For all $1 \leq j < +\infty$ and $k \geq 1$, $\frac{N_{n,(j,k)}}{n} \rightarrow \eta'_{(j,k)}$, almost surely as $n \rightarrow +\infty$.*

5 Open Problems

A challenging open problem is to give an exact quantitative description of the dynamics of the innovation-pays-off phase. Our results imply that a constant fraction of the links “escapes at infinity.” But we know little about the transient behavior in this regime. How are the links distributed among the highest fitnesses present in the system at any given time? At what rate are new nodes with higher fitnesses taking over? How does the transient behavior depend on the fitness distribution? This could have important practical implications.

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²Strictly speaking, the equality case belongs to the fit-get-rich phase since Equation (6) has a solution, namely h ; nevertheless we include it in this section because its proof is more similar to the innovation-pays-off phase.

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A Analysis of Bounded Discrete Fitness Distributions

Proof of Proposition 1: We complete the proof of Proposition 1 by computing the largest positive eigenvalue λ_1 and the corresponding left eigenvector v_1 of the matrix A . Because the ξ_i 's are deterministic, it follows that $A_{ij} = \xi_{i,j}$ for all $1 \leq i, j \leq q$. To compute λ_1 we first compute the corresponding *right* eigenvector. Note that

$$\sum_{j=1}^q \xi_{i,j} = 2,$$

for all $1 \leq i \leq q$ and therefore u_1 is $(1, \dots, 1)$ (up to a constant factor) and $\lambda_1 = 2$. The left eigenvector v_1 must satisfy,

$$\sum_{i=1}^q (v_1)_i = 1,$$

by convention, as well as,

$$\sum_{i=2}^q (v_1)_i = 2(v_1)_1,$$

which with the previous equation implies $(v_1)_1 = 1/3$. Also, for $2 \leq l \leq q-1$,

$$l(v_1)_{l-1} - l(v_1)_l = 2(v_1)_l,$$

or,

$$\frac{(v_1)_l}{(v_1)_{l-1}} = \frac{l}{l+2}.$$

Therefore,

$$(v_1)_k = \prod_{i=1}^k \frac{l}{l+2}.$$

Finally, by Theorem 2, we get

$$\frac{L_{n,k}}{n} = \frac{X_{n,k}}{kn} \rightarrow \frac{2(v_1)_k}{k} = \mu_k.$$

almost surely as $n \rightarrow +\infty$. ■

Proof of Proposition 2: We complete the proof of Proposition 2 by computing the largest positive eigenvalue λ_1 and the corresponding left eigenvector v_1 of the matrix A which has the following form

$$A_{ij} = f_i(q_j + \mathbf{1}_{\{i=j\}}).$$

We compute the corresponding λ_1, v_1 . For all $1 \leq j \leq J$, v_1 must satisfy

$$q_j \sum_{i=1}^J f_i(v_1)_i + f_j(v_1)_j = \lambda_1(v_1)_j. \quad (11)$$

By the convention

$$a \cdot v_1 = 1 \quad \Leftrightarrow \quad \sum_{i=1}^J f_i(v_1)_i = 1, \quad (12)$$

it follows that, for all $1 \leq j \leq J$,

$$(v_1)_j = \frac{q_j}{\lambda_1 - f_j}.$$

Plugging back into (12), we get

$$\sum_{j=1}^J \frac{f_j q_j}{\lambda_1 - f_j} = 1.$$

Therefore, $\lambda_1 = \lambda_0$ and $(v_1)_j = (\lambda_1)^{-1} \nu_j$ for all $1 \leq j \leq J$. The result follows by Theorem 2. ■

Proof of Proposition 3: Fix $1 \leq j \leq J$ and $k \geq 1$. Set $r = k + 1$ and $q = rJ$. Consider the following urn process which is a combination of those in Propositions 1 and 2. We now have a bin—indexed (i, l) —for each fitness f_i and each degree l up to k . The number of balls in bin (i, l) at time n is denoted $X_{n,(i,l)}$. The urn process is defined so that $X_{n,(i,l)} = lN_{n,(i,l)}$ (see below). Also, for each i , the bin (i, r) counts all the links attached to a vertex of fitness f_i and degree more than k , that is we have

$$X_{n,(i,r)} = \sum_{l \geq k+1} lN_{n,(i,l)}.$$

The activity of bin (i, l) is $a_{(i,l)} = f_i$. Say at step n we pick a ball from bin (i, l) with $1 < l < r$. Then,

1. we choose a fitness, say i' , according to \mathcal{Q} ;
2. we add one ball to bin $(i', 1)$;
3. we remove l balls from bin (i, l) ;
4. we add $l + 1$ balls to bin $(i, l + 1)$.

The cases $l = 1, r$ are handled similarly (see Proposition 1).

We compute matrix A . Let (i, l) be such that $1 < l < r$. Then row (i, l) of A is

$$A_{(i,l),(i',l')} = \begin{cases} -f_i l, & i' = i, l' = l \\ f_i(l + 1), & i' = i, l' = l + 1 \\ f_i q_{i'}, & l' = 1 \\ 0, & \text{o.w.} \end{cases}$$

For $l = 1$, we get

$$A_{(i,1),(i',l')} = \begin{cases} f_i(-1 + q_i), & i' = i, l' = 1 \\ 2f_i, & i' = i, l' = 2 \\ f_i q_{i'}, & i' \neq i, l' = 1 \\ 0, & \text{o.w.} \end{cases}$$

and, for $l = r$,

$$A_{(i,r),(i',l')} = \begin{cases} f_i, & i' = i, l' = r \\ f_i q_{i'}, & l' = 1 \\ 0, & \text{o.w.} \end{cases}$$

We compute the corresponding λ_1, u_1, v_1 . Consider the following guess for u_1

$$(u_1)_{(i,l)} = \frac{f_i}{\lambda_0 - f_i},$$

for all $1 \leq i \leq J$ and $1 \leq l \leq r$ where λ_0 is defined in (3). Then we have for $1 \leq i \leq J$ and $1 \leq l \leq q$,

$$\begin{aligned}
\sum_{(i',l')} A_{(i,l),(i',l')}(u_1)_{(i',l')} &= f_i \sum_{i'} \frac{f_{i'} q_{i'}}{\lambda_0 - f_{i'}} + \frac{f_i^2}{\lambda_0 - f_i} \\
&= f_i + \frac{f_i^2}{\lambda_0 - f_i} \\
&= \frac{f_i}{\lambda_0 - f_i} (\lambda_0 - f_i + f_i) \\
&= \lambda_0 (u_1)_{(i,l)},
\end{aligned}$$

where we used (3). Hence, the Perron-Frobenius eigenvalue is $\lambda_1 = \lambda_0$ and the corresponding right eigenvector is u_1 as above.

It remains to compute v_1 . Define the auxiliary vector

$$(\tilde{v}_1)_i = \sum_{l=1}^r (v_1)_{(i,l)},$$

for $1 \leq i \leq J$. Then, by looking at column $(i, 1)$ of A , we must have

$$q_i \sum_{i'=1}^J f_{i'} (\tilde{v}_1)_{i'} - f_i (v_1)_{(i,1)} = \lambda_1 (v_1)_{(i,1)}, \quad (13)$$

for all $1 \leq i \leq J$. From column (i, r) we get

$$f_i (r(v_1)_{(i,r-1)} + (v_1)_{i,r}) = \lambda_1 (v_1)_{(i,r)}. \quad (14)$$

Finally, for $1 < l < r$, column (i, l) gives

$$f_i (l(v_1)_{(i,l-1)} - l(v_1)_{i,l}) = \lambda_1 (v_1)_{(i,l)}. \quad (15)$$

Summing (13), (14), and (15), we obtain

$$q_i \sum_{i'=1}^J f_{i'} (\tilde{v}_1)_{i'} + f_i (\tilde{v}_1)_i = \lambda_1 (\tilde{v}_1)_i.$$

This is identical to (11) from Proposition 2 and therefore

$$(\tilde{v}_1)_i = \frac{q_i}{\lambda_1 - f_i},$$

for all $1 \leq i \leq J$. Also, from (15), for $1 < l < r$, we get

$$\frac{(v_1)_{(i,l)}}{(v_1)_{(i,l-1)}} = \frac{l}{l + \lambda_1 f_i^{-1}}.$$

By our convention,

$$\sum_{i'=1}^J f_{i'} (\tilde{v}_1)_{i'} = 1,$$

we get from (13),

$$(v_1)_{(i,1)} = \frac{q_i}{\lambda_1 + f_i}.$$

From Theorem 2, we derive

$$\frac{N_{n,(j,k)}}{n} = \frac{X_{n,(j,k)}}{kn} \rightarrow \frac{\lambda_1 (v_1)_{(j,k)}}{k} = \eta_{(j,k)},$$

almost surely as $n \rightarrow +\infty$. This concludes the proof. ■

B Analysis of Countable Discrete Fitness Distributions

B.1 Coupling

We derive bounds on $N_{n,(j,k)}$, defined to be the number of vertices of fitness f_j and degree k at time n in the $(\mathcal{F}, \mathcal{Q})$ -chain of Section 4. Fix I and let $\overline{N}_{n,(j,k)}, \underline{N}_{n,(j,k)}$ be the corresponding variables for the chains $(\overline{\mathcal{F}}, \mathcal{Q})$ and $(\underline{\mathcal{F}}, \mathcal{Q})$ of Section 4.1 defined by the I -truncations of \mathcal{F} . Since the latter have finite fitness distributions, we can apply Proposition 3. Let $\overline{\eta}_{(j,k)}$ and $\underline{\eta}_{(j,k)}$ be the almost sure limits of $n^{-1}\overline{N}_{n,(j,k)}$ and $n^{-1}\underline{N}_{n,(j,k)}$. For the full coupling, we also need the degree tails for a fixed fitness. Let

$$T_{n,(j,k)} = \sum_{k' \geq k} k' N_{n,(j,k')},$$

and similarly for $\overline{T}_{n,(j,k)}$ and $\underline{T}_{n,(j,k)}$. Also, let

$$\overline{\tau}_{(j,k)} = \sum_{k' \geq k} k' \overline{\eta}_{(j,k')},$$

and similarly for $\underline{\tau}_{(j,k)}$. These are well-defined because the partial sums are increasing and bounded by 2 (see the proof of Proposition 3). The following lemma provides a full coupling of the chains $(\mathcal{F}, \mathcal{Q})$, $(\overline{\mathcal{F}}, \mathcal{Q})$ and $(\underline{\mathcal{F}}, \mathcal{Q})$.

Lemma 2 (Coupling: Full Analysis) *For all $1 \leq j \leq I$ and $k \geq 1$, it holds almost surely that*

$$\limsup_{n \rightarrow +\infty} \frac{T_{n,(j,k)}}{n} \leq \overline{\tau}_{(j,k)} \quad \text{and} \quad \liminf_{n \rightarrow +\infty} \frac{T_{n,(j,k)}}{n} \geq \underline{\tau}_{(j,k)}.$$

Proof of Lemma 2: As in Lemma 1, we couple the $(\mathcal{F}, \mathcal{Q})$ -chain and the truncations. We use the notation of Lemma 1. Also, for $k \geq 1$, let D_n be the degree of the vertex picked at time n in the $(\mathcal{F}, \mathcal{Q})$ -chain (and similarly for $\overline{D}_n, \underline{D}_n$). For $1 \leq i \leq I$ and $k \geq 1$, let $\sigma_{n-1,(i,k)}$ be the probability of the event $\{F_n = f_i, D_n \geq k\}$ given the state after time $n-1$ in the $(\mathcal{F}, \mathcal{Q})$ -chain (and similarly for $\overline{\sigma}_n, \underline{\sigma}_n$). We require the following conditions to be satisfied:

1. For all $n \geq 1$,

$$\begin{aligned} \overline{F}_n &\leq F_n \leq \underline{F}_n, \\ \overline{F}'_n &\leq F'_n \leq \underline{F}'_n. \end{aligned}$$

2. For all $n \geq 1$ and all $1 \leq i \leq I$,

$$\underline{M}_{n,i} \leq M_{n,i} \leq \overline{M}_{n,i}.$$

3. For all $n \geq 1$ and all $1 \leq i \leq I$,

$$\underline{\rho}_{n,i} \leq \rho_{n,i} \leq \overline{\rho}_{n,i}.$$

4. For all $n \geq 1$, $1 \leq i \leq I$, and $k \geq 1$,

$$\underline{T}_{n,(i,k)} \leq T_{n,(i,k)} \leq \overline{T}_{n,(i,k)}.$$

5. For all $n \geq 1$, $1 \leq i \leq I$, and $k \geq 1$,

$$\underline{\sigma}_{n,(i,k)} \leq \sigma_{n,(i,k)} \leq \overline{\sigma}_{n,(i,k)}.$$

These conditions are somewhat redundant but we keep all of them for clarity. In particular, note that 3. follows from 1. and 2., that 5. follows from 1. and 4., and that 2. and 3. are special cases of 4. and 5. Assume these conditions hold up to $n-1$. Our step-by-step coupling has two parts. First, we pick the fitnesses $\overline{F}_n, F_n, \underline{F}_n, \overline{F}'_n, F'_n, \underline{F}'_n$ using the scheme described in the proof of Lemma 1. We then pick the degrees $\underline{D}_n, D_n, \overline{D}_n$ by picking a *single* uniform random variable in $[0, 1]$ and “inverting” simultaneously the tails $\{\underline{\sigma}_{n,(\underline{F}_n,k)}\}_{k \geq 1}$, $\{\sigma_{n,(F_n,k)}\}_{k \geq 1}$, and $\{\overline{\sigma}_{n,(\overline{F}_n,k)}\}_{k \geq 1}$. (This is sometimes called the “inverse transform sampling method”.) It is easy to check that all conditions are then satisfied at time n . ■

B.2 Fit-Get-Richer Phase

Proof of Theorem 3: We only need to consider the case $J = +\infty$. Fix $1 \leq j < +\infty$ and $k \geq 1$. Let $1 \leq I < +\infty$ and consider once again the I -truncations of the $(\mathcal{F}, \mathcal{Q})$ -chain. Let $\underline{\nu}_j^I, \overline{\nu}_j^I, \underline{\eta}_{(j,k)}^I, \overline{\eta}_{(j,k)}^I, \underline{\tau}_{(j,k)}^I, \overline{\tau}_{(j,k)}^I$ be as in Lemmas 1, 2 (we now indicate the dependence on I because we will need to take $I \rightarrow +\infty$). Similarly, let $\underline{\lambda}_0^I$ and $\overline{\lambda}_0^I$ be the largest solution to (6) for the lower and upper truncations. By the coupling lemmas, it suffices to prove

$$\underline{\lambda}_0^I, \overline{\lambda}_0^I \rightarrow \lambda_0, \quad (16)$$

as $I \rightarrow +\infty$. Indeed, in that case

$$\underline{\nu}_j^I, \overline{\nu}_j^I \rightarrow \nu_j,$$

as $I \rightarrow +\infty$, which implies

$$\frac{M_{n,j}}{n} \rightarrow \nu_j,$$

by Lemma 1. Also, for all $l \leq k$,

$$\underline{\eta}_{(j,l)}^I, \overline{\eta}_{(j,l)}^I \rightarrow \eta_{(j,l)},$$

as $I \rightarrow +\infty$, which implies

$$\underline{\tau}_{(j,k)}^I = \underline{\nu}_j^I - \sum_{l \leq k} l \underline{\eta}_{(j,l)}^I \rightarrow \nu_j - \sum_{l \leq k} l \eta_{(j,l)},$$

as $I \rightarrow +\infty$, and similarly for $\overline{\tau}_{(j,k)}^I$. This also holds for $k-1$ so that, by Lemma 2, we have

$$\frac{N_{n,(j,k)}}{n} \rightarrow \eta_{(j,k)},$$

almost surely as $n \rightarrow +\infty$.

It remains to prove (16). We argue about $\overline{\lambda}_0^I$. The proof for $\underline{\lambda}_0^I$ is similar and is omitted. Let

$$S(\lambda) := \sum_{i=1}^{+\infty} \frac{f_i q_i}{\lambda - f_i}, \quad \overline{S}^I(\lambda) := \sum_{i=1}^{+\infty} \frac{\overline{f}_i q_i}{\lambda - \overline{f}_i}.$$

Note that for $\lambda' > \lambda > h$, we have

$$S(\lambda'), S(\lambda) \leq \frac{h}{\lambda - h}, \quad S(\lambda') < S(\lambda),$$

and

$$|S(\lambda') - S(\lambda)| \leq \frac{|\lambda' - \lambda| h}{|\lambda' - h| |\lambda - h|}.$$

Therefore, S is continuous and strictly decreasing on $\{\lambda > h\}$. Also, by definition of $\overline{\mathcal{F}}$, we have

$$\overline{R}^I(\lambda) := S(\lambda) - \overline{S}^I(\lambda) = \sum_{i=I+1}^{+\infty} \frac{f_i q_i}{\lambda - f_i}.$$

Therefore, for $\lambda > h$,

$$\left| \overline{R}^I(\lambda) \right| \leq \frac{h}{\lambda - h} \sum_{i=I+1}^{+\infty} q_i \rightarrow 0,$$

as $I \rightarrow +\infty$. Hence, for all $\varepsilon > 0$ (small enough),

$$\lim_{I \rightarrow \infty} \overline{S}^I(\lambda_0 + \varepsilon) = S(\lambda_0 + \varepsilon) < 1, \quad \lim_{I \rightarrow \infty} \overline{S}^I(\lambda_0 - \varepsilon) = S(\lambda_0 - \varepsilon) > 1,$$

so that eventually

$$\lambda_0 - \varepsilon \leq \overline{\lambda}_0^I \leq \lambda_0 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have (16). ■

B.3 Innovation-Pays-Off Phase

Example 1 *The case $J < +\infty$ always satisfies (7). Indeed, in that case,*

$$\sum_{j=1}^J \frac{f_j q_j}{h - f_j} = +\infty. \tag{17}$$

Likewise, when $J = +\infty$ and the fitness supremum h is attained, we also get (17).

Example 2 *Consider the case $f_j = 1 - j^{-1}$ for all $j \geq 1$ and*

$$q_j = \frac{j^{2+\theta}}{\zeta(2+\theta)},$$

where ζ is the Riemann zeta function. In particular, by definition, $\sum_{j \geq 1} q_j = 1$. Here, $h = 1$ is not attained. We now compute the sum in (8). We have

$$\begin{aligned} \sum_{j=1}^J \frac{f_j q_j}{h - f_j} &= \sum_{j \geq 1} \frac{(1 - j^{-1}) \zeta^{-1}(2 + \theta) j^{-2-\theta}}{j^{-1}} \\ &= \zeta^{-1}(2 + \theta) \left(\sum_{j \geq 1} j^{-1-\theta} - \sum_{j \geq 1} j^{-2-\theta} \right) \\ &= \frac{\zeta(1 + \theta) - \zeta(2 + \theta)}{\zeta(2 + \theta)}. \end{aligned}$$

One can check that the last line is < 1 when $\theta > 1$. This example can be seen as a “discretization” of the example given in [4].

Proof of Theorem 4: We use the notations of Theorem 3. Similarly to Theorem 3, it suffices to prove

$$\underline{\lambda}_0^I, \overline{\lambda}_0^I \rightarrow h, \tag{18}$$

as $I \rightarrow +\infty$. Let

$$h^I = \sup_{j \leq I} f_j.$$

By a remark above the statement of the Theorem, we know that $h^I < h$ and $h^I \rightarrow h$ as $I \rightarrow +\infty$.

We first argue about $\bar{\lambda}_0^I$. Note that $\bar{\lambda}_0^I > h^I$. Also, $\bar{S}^I(h) < S(h) \leq 1$ and therefore $\bar{\lambda}_0^I \leq h$. That implies $\bar{\lambda}_0^I \rightarrow h$.

Now consider the case of $\underline{\lambda}_0^I$. Let

$$\underline{R}^I(\lambda) := S(\lambda) - \underline{S}^I(\lambda).$$

We have, for all $\varepsilon > 0$,

$$S(h + \varepsilon) < S(h) \leq 1,$$

and

$$|\underline{R}^I(h + \varepsilon)| \leq \frac{h}{\varepsilon} \sum_{i=I+1}^{+\infty} q_i \rightarrow 0,$$

as $I \rightarrow +\infty$. Hence, for all $\varepsilon > 0$,

$$\lim_{I \rightarrow \infty} \underline{S}^I(h + \varepsilon) = S(h + \varepsilon) < 1,$$

so that eventually

$$h^I < \underline{\lambda}_0^I \leq h + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\underline{\lambda}_0^I \rightarrow h$ as $I \rightarrow +\infty$. ■

B.4 Unbounded Countable Case

Assume $h = \sup_{j \geq 1} f_j = +\infty$, i.e. the set of fitnesses is unbounded. In that case, the lower bounds in the coupling lemmas cannot be used but it turns out that the upper bounds suffice to characterize the limit behavior of the process.

Theorem 5 (Discrete Case: Unbounded Fitness) *Assume $\sup_{j \geq 1} f_j = +\infty$. Then it holds that*

1. For all $1 \leq j < +\infty$,

$$\frac{M_{n,j}}{n} \rightarrow q_j,$$

almost surely as $n \rightarrow +\infty$.

2. For all $1 \leq j < +\infty$ and $k > 1$,

$$\frac{T_{n,(j,k)}}{n} \rightarrow 0,$$

almost surely as $n \rightarrow +\infty$.

Proof: Fix $1 \leq j < +\infty$ and $k > 1$. We use the upper bounds in the coupling Lemmas 1 and 2. We use the notations of Theorems 3 and 4. We have that $h^I \rightarrow +\infty$ and therefore $\bar{\lambda}_0^I > h^I \rightarrow +\infty$. Therefore, plugging into the equations for \bar{v}_j^I and $\bar{\tau}_{(j,k)}^I = \bar{v}_j^I - \sum_{l \leq k} l \bar{\eta}_{(j,l)}^I$, we get

$$\limsup_{n \rightarrow \infty} \frac{M_{n,j}}{n} \leq q_j,$$

and

$$\limsup_{n \rightarrow \infty} \frac{T_{n,(j,k)}}{n} \leq 0,$$

almost surely. We get 2. immediately. To get 1., consider the following chain $\{X_{n,i}\}_{n,i \geq 0}$. Pick a fitness say F_0 according to \mathcal{Q} and let $X_0 = e_{F_0}$. Then at each time step, pick a fitness F_n according to \mathcal{Q} and set $X_n = X_{n-1} + e_{F_n}$. This chain can clearly be coupled with the $(\mathcal{F}, \mathcal{Q})$ -chain in such a way that $M_n \geq X_n$ for all n . Now it is easy to see that $X_{n,j} \rightarrow q_j$ as $n \rightarrow +\infty$, and therefore

$$\liminf_{n \rightarrow \infty} \frac{M_{n,j}}{n} \geq q_j.$$

This concludes the proof. ■

C Analysis of Continuous Fitness Distributions

In this section, we analyze the preferential attachment scheme under continuous fitness distributions. Let $h < +\infty$ —the unbounded case is treated in Appendix C.4—and let $g : [0, h] \rightarrow \mathbb{R}_+$ be a continuous density function. Consider the preferential attachment process with $\mathcal{F} = [0, h]$ and \mathcal{Q} the distribution defined by g . The dynamical behavior parallels the one observed in the discrete case, namely

1. the fit-get-richer scenario taking place when $\int_0^h \frac{xg(x)}{h-x} dx \geq 1$,
2. the innovation-pays-off scenario taking place when $\int_0^h \frac{xg(x)}{h-x} dx < 1$.

The analysis requires a more sophisticated coupling argument than that for the discrete case described in Section 4.

C.1 Coupling

We discretize the $(\mathcal{F}, \mathcal{Q})$ -chain in the following way, which lets us bound the relevant quantities from below only. It will turn out that the lower bound is sufficient for our purposes. Fix $1 < I < +\infty$, an integer with $\varepsilon = h\frac{1}{I}$. For $1 \leq i \leq I$, let

$$\begin{aligned} \bar{f}_i &= h \frac{i}{I}, \\ \underline{f}_i &= h \frac{i-1}{I}, \end{aligned}$$

and

$$\tilde{q}_i = \int_{\underline{f}_i}^{\bar{f}_i} g(x) dx.$$

Denote $\tilde{\mathcal{Q}}$ the distribution over $\{1, 2, \dots, I\}$ defined by $\{\tilde{q}_i\}_{i=1}^I$. For reasons that will be clear in Section C.3, we allow $\int_0^h g(x) dx < 1$. Consider the following finite balls-in-bins process with $q = I+1$ bins. The activities are

$$a_i = \begin{cases} \bar{f}_i, & \text{if } i \leq I, \\ h, & \text{if } i = I+1. \end{cases}$$

For the initial load, let $1 \leq i^* \leq I$ be picked according to $\tilde{\mathcal{Q}}$ and pose

$$X_{0,i} = \begin{cases} 2, & \text{if } i = i^*, \\ 0, & \text{o.w.} \end{cases}$$

The update vectors are defined as follows for $1 \leq i \leq I$: pick i^* according to $\tilde{\mathcal{Q}}$ (set $i^* = +\infty$ with probability $1 - G$ where $G = \int_0^h g(x)dx$), let $\gamma_i = 1$ with probability $\underline{f}_i/\bar{f}_i$ and 0 o.w., and set

$$\xi_{i,i'} = \gamma_i \mathbf{1}_{\{i'=i\}} + \begin{cases} 1, & \text{if } i' = i^*, \\ 1, & \text{if } i' = I+1, \gamma_i = 0, \\ 0, & \text{o.w.} \end{cases}$$

and

$$\xi_{I+1,i'} = \begin{cases} 1, & \text{if } i' = i^*, \\ 1, & \text{if } i' = I+1, \\ 0, & \text{o.w.} \end{cases}$$

Because this chain is not exactly of the type described in Section 3, we cannot appeal directly to Proposition 2. Therefore, we give a separate analysis here. Let $\tilde{\lambda}_0 > h - \varepsilon$ be a solution to

$$\sum_{j=1}^I \frac{\bar{f}_j \tilde{q}_j}{\tilde{\lambda}_0 - \underline{f}_j} + h \frac{(1+G)\tilde{\lambda}_0^{-1}\varepsilon}{\tilde{\lambda}_0 - (h - \varepsilon)} = 1. \quad (19)$$

By monotonicity, it is clear that there is a unique such solution. For $1 \leq j \leq I$, let

$$\tilde{\nu}_j = \tilde{\lambda}_0 \frac{\tilde{q}_j}{\tilde{\lambda}_0 - \underline{f}_j},$$

and

$$\tilde{\nu}_{I+1} = \tilde{\lambda}_0 \frac{(1+G)\tilde{\lambda}_0^{-1}\varepsilon}{\tilde{\lambda}_0 - (h - \varepsilon)}.$$

Note that

$$\begin{aligned} \left(1 + \frac{\varepsilon}{\lambda_0}\right) \sum_{j=1}^{I+1} \tilde{\nu}_j &= \sum_{j=1}^I \tilde{q}_j + (1+G)\varepsilon + \sum_{j=1}^I \frac{\bar{f}_j \tilde{q}_j}{\tilde{\lambda}_0 - \underline{f}_j} + h \frac{(1+G)\tilde{\lambda}_0^{-1}\varepsilon}{\tilde{\lambda}_0 - (h - \varepsilon)} \\ &= G + (1+G) \frac{\varepsilon}{\lambda_0} + 1 \\ &= (1+G) \left(1 + \frac{\varepsilon}{\lambda_0}\right), \end{aligned}$$

so that

$$\sum_{j=1}^{I+1} \tilde{\nu}_j = 1 + G. \quad (20)$$

We prove the following.

Lemma 3 (Discretization) *For all $1 \leq j \leq I+1$,*

$$\frac{X_{n,j}}{n} \rightarrow \tilde{\nu}_j,$$

almost surely as $n \rightarrow +\infty$.

Proof: The matrix A has the following form: for $1 \leq i \leq I$, $1 \leq j \leq I+1$,

$$A_{ij} = \bar{f}_i \left(\tilde{q}_j \mathbf{1}_{\{j \leq I\}} + \frac{\underline{f}_j}{\bar{f}_i} \mathbf{1}_{\{j=i\}} + \frac{\varepsilon}{\bar{f}_i} \mathbf{1}_{\{j=I+1\}} \right),$$

and for $i = I + 1$

$$A_{I+1,j} = h(\tilde{q}_j \mathbf{1}_{\{j \leq I\}} + \mathbf{1}_{\{j=I+1\}}).$$

We compute the corresponding λ_1, v_1 . Note that by Theorem 2 and the law of large numbers, it is clear that

$$\sum_{i=1}^{I+1} \lambda_1(v_1)_i = 1 + G. \quad (21)$$

For all $1 \leq j \leq I$, v_1 must satisfy

$$\tilde{q}_j \sum_{i=1}^q a_i(v_1)_i + \bar{f}_j \frac{f_j}{\bar{f}_j}(v_1)_j = \lambda_1(v_1)_j.$$

By the convention

$$\sum_{i=1}^q a_i(v_1)_i = 1, \quad (22)$$

it follows that for all $1 \leq j \leq I$

$$(v_1)_j = \frac{\tilde{q}_j}{\lambda_1 - \bar{f}_j}.$$

Also for $i = I + 1$, we must have

$$\sum_{i=1}^I \bar{f}_i \frac{\varepsilon}{\bar{f}_i}(v_1)_i + h(v_1)_{I+1} = \varepsilon((1+G)\lambda_1^{-1} - (v_1)_{I+1}) + h(v_1)_{I+1} = \lambda_1(v_1)_{I+1},$$

where we have used (21). Therefore,

$$(v_1)_{I+1} = \frac{(1+G)\lambda_1^{-1}\varepsilon}{\lambda_1 - (h - \varepsilon)}.$$

Plugging back into (22), we get

$$\sum_{j=1}^I \frac{\bar{f}_j \tilde{q}_j}{\lambda_1 - \bar{f}_j} + h \frac{(1+G)\lambda_1^{-1}\varepsilon}{\lambda_1 - (h - \varepsilon)} = 1.$$

Therefore, $\lambda_1 = \tilde{\lambda}_0$ and $(v_1)_j = (\lambda_1)^{-1} \tilde{\nu}_j$ for all $1 \leq j \leq q$. The result follows by Theorem 2. ■

Consider again the $(\mathcal{F}, \mathcal{Q})$ -chain. For $n \geq 0$ and $1 \leq j \leq I$, let $M_{n,j}$ be the number of edges with an endpoint of fitness in $(\underline{f}_j, \bar{f}_j)$ (counting twice edges with two endpoints of fitness in $(\underline{f}_j, \bar{f}_j)$). Then we have the following.

Lemma 4 (Coupling: Continuous Case) *For all $1 \leq j \leq I$, it holds that*

$$\liminf_{n \rightarrow +\infty} \frac{M_{n,j}}{n} \geq \tilde{\nu}_j,$$

almost surely.

Proof: This proof is similar to the proof of Lemma 1. Consider the $(\mathcal{F}, \mathcal{Q})$ -chain. At step $n \geq 1$, we first pick a vertex according to weighted preferential attachment. Let F_n be the fitness of the chosen vertex, and denote $\rho_{n-1,i}$ the probability that $F_n \in (\underline{f}_i, \bar{f}_i)$ given the state after time $n - 1$. Secondly, we add a new vertex with fitness according to \mathcal{Q} . Let F'_n be the fitness of this new vertex.

Similarly for the discretized chain, we first pick a bin i by weighted preferential attachment and then an i^* according to \tilde{Q} . We also pick γ_i a Bernoulli($\underline{f}_i/\bar{f}_i$). We let

$$\underline{F}_n = \begin{cases} \bar{f}_i, & \text{if } \gamma_i = 1, \\ h, & \text{if } \gamma_i = 0. \end{cases} \quad \text{and} \quad \underline{F}'_n = \begin{cases} \bar{f}_{i^*}, & \text{if } i^* \leq I, \\ +\infty, & \text{if } i^* = +\infty, \end{cases}$$

We denote $\underline{\rho}_{n-1,i}$ the probability that $\underline{F}_n = \bar{f}_i$ and $\gamma_i = 1$ given the state after time $n-1$ ³. We couple the two chains so as to preserve the following conditions:

1. For all $n \geq 1$,

$$F_n \leq \underline{F}_n,$$

and

$$F'_n \leq \underline{F}'_n.$$

2. For all $n \geq 1$ and all $1 \leq i \leq I$,

$$\underline{M}_{n,i} \leq M_{n,i}.$$

3. For all $n \geq 1$ and all $1 \leq i \leq I$,

$$\underline{\rho}_{n,i} \leq \rho_{n,i}.$$

Note that 3. follows easily from 1., 2. and the definition of γ_i . In fact, the reason for using the “rejection” variable γ_i is to keep $\underline{\rho}_{n,i}$ small by making its numerator small—with a contribution of only \underline{f}_i —while preserving a large denominator. Here is how our coupling works. In the initial configuration, the $(\mathcal{F}, \mathcal{Q})$ -chain has one vertex with a self-loop and fitness $F_0 = f_i$, where f_i is picked according to \mathcal{Q} ; the discretized chain can be coupled so that two balls are added to a bin with activity $\underline{F}_0 = \bar{f}_i$ with probability $\underline{f}_i/\bar{f}_i$ and $\underline{F}_0 = h$ with probability $1 - \underline{f}_i/\bar{f}_i$. Therefore the conditions are satisfied at time 0 by construction. Assume Conditions 1., 2., and 3. are satisfied at time $n-1$; we will show then that they are also satisfied at time n . First, consider picking fitness for the new vertex. In the $(\mathcal{F}, \mathcal{Q})$ -chain, $F'_n = f_i$, where f_i is picked according to \mathcal{Q} ; the choice of the discretized chain can be coupled so that $\underline{F}'_n = \bar{f}_i$. Therefore, $F'_n \leq \underline{F}'_n$. Now consider the step of choosing an old vertex. By 3., it is clear how to choose the F ’s so as to satisfy 1. and 2. Indeed, proceed as follows:

- With probability $\sum_{i=1}^I \underline{\rho}_{n-1,i}$, pick a bin according to $\{\underline{\rho}_{n-1,i}\}_{i=1}^I$ in the discretized chain, say i , and pick a fitness according to weighted preferential attachment restricted to $(\underline{f}_i, \bar{f}_i)$ for the $(\mathcal{F}, \mathcal{Q})$ -chain (the interval $(\underline{f}_i, \bar{f}_i)$ is nonempty by 2.);
- With remaining probability, pick bin $I+1$ for the discretized chain, pick an interval according to $\{(\rho_{n-1,i} - \underline{\rho}_{n-1,i})\}_{i=1}^I$, say $(\underline{f}_i, \bar{f}_i)$, and pick a fitness according to weighted preferential attachment restricted to $(\underline{f}_i, \bar{f}_i)$ for the $(\mathcal{F}, \mathcal{Q})$ -chain.

This concludes the proof. ■

C.2 Fit-Get-Richer Phase

Assume the density g is defined on $[0, h]$ with $h < +\infty$ and assume further that $g(x) > 0$ for all $x \in (0, h)$ (we allow 0 at the endpoints). In this section, we consider the case

$$\int_0^h \frac{xg(x)}{h-x} dx \geq 1. \tag{23}$$

The remaining cases are treated in the following two subsections.

³ The specification that $\gamma_i = 1$ is relevant only in the case $i = I$.

Example 3 An important special case of (23) is when $g(h) > 0$. Indeed, take any $\delta > 0$ small and let $\delta' = \inf_{x \in [h-\delta, h]} g(x)$. Note that $\delta' > 0$ by assumption. Then,

$$\begin{aligned} \int_0^h \frac{xg(x)}{h-x} dx &\geq \int_{h-\delta}^h \frac{xg(x)}{h-x} dx \\ &\geq (h-\delta)\delta' \int_{h-\delta}^h \frac{1}{h-x} dx \\ &\geq (h-\delta)\delta' \int_0^\delta \frac{1}{y} dy \\ &= +\infty \\ &\geq 1. \end{aligned}$$

This example will turn out to be useful in Section C.3.

By (23) and monotonicity, there exists a solution $\lambda_0 \geq h$ to

$$\int_0^h \frac{xg(x)}{\lambda_0 - x} dx = 1. \quad (24)$$

For $0 \leq a < b \leq h$, let

$$\nu_{[a,b]} = \lambda_0 \int_a^b \frac{g(x)}{\lambda_0 - x} dx.$$

Note in particular that

$$\nu_{[0,h]} = \int_0^h (\lambda_0 - x) \frac{g(x)}{\lambda_0 - x} dx + \int_0^h \frac{xg(x)}{\lambda_0 - x} dx = 1 + G, \quad (25)$$

as one would expect (but see Section C.3 below). Also, for $n \geq 0$, let $M_{n,[a,b]}$ be the number of edges with an endpoint of fitness in $[a, b]$ (counting twice edges with two endpoints of fitness in $[a, b]$).

We prove the following.

Theorem 6 (Continuous Case: Fit-Get-Richer Phase) Assume g is defined on $[0, h]$ with $h < +\infty$ and assume further that $g(x) > 0$ for all $x \in (0, h)$ and

$$\int_0^h \frac{xg(x)}{h-x} dx \geq 1.$$

Then it holds that for all $0 \leq a < b \leq h$,

$$\frac{M_{n,[a,b]}}{n} \rightarrow \nu_{[a,b]},$$

almost surely as $n \rightarrow +\infty$.

Proof: Note that the law of large numbers implies

$$\frac{M_{n,[0,h]}}{n} \rightarrow 1 + G,$$

almost surely as $n \rightarrow +\infty$, so that by (25) it suffices to show that

$$\liminf_{n \rightarrow +\infty} \frac{M_{n,[a,b]}}{n} \geq \nu_{[a,b]},$$

almost surely for all $0 \leq a < b \leq h$.

Let $1 \leq I < +\infty$ and consider once again the discretization of the $(\mathcal{F}, \mathcal{Q})$ -chain. Let $\tilde{\nu}_j^I$ be as in Lemma 4 (we now indicate the dependence on I because we will need to take $I \rightarrow +\infty$). Similarly, let $\tilde{\lambda}_0^I$ be as in (19). Fix $0 \leq a < b \leq h$. Let \mathcal{K}^I be the largest subset of $\{1, \dots, I\}$ such that

$$\bigcup_{i \in \mathcal{K}^I} (\underline{f}_i^I, \bar{f}_i^I) \subseteq [a, b].$$

By the coupling lemma, we have

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{M_{n, [a, b]}}{n} &\geq \sum_{i \in \mathcal{K}^I} \tilde{\nu}_i^I \\ &= \sum_{i \in \mathcal{K}^I} \frac{\tilde{q}_i^I}{\tilde{\lambda}_0^I - \underline{f}_i^I} \\ &\geq \sum_{i \in \mathcal{K}^I} \int_{\underline{f}_i^I}^{\bar{f}_i^I} \frac{g(x)}{\tilde{\lambda}_0^I + \varepsilon - x} dx \\ &\geq \int_{a+\varepsilon}^{b-\varepsilon} \frac{g(x)}{\tilde{\lambda}_0^I + \varepsilon - x} dx. \end{aligned}$$

Since $\varepsilon = 1/I$ goes to 0 as $I \rightarrow +\infty$, it suffices to prove

$$\tilde{\lambda}_0^I \rightarrow \lambda_0, \tag{26}$$

as $I \rightarrow +\infty$.

We first show that $\tilde{\lambda}_0 > \lambda_0 - \varepsilon$. Indeed, assume $\tilde{\lambda}_0^I = \lambda_0 - \varepsilon$. Then, the sum in (19) satisfies

$$\begin{aligned} \sum_{j=1}^I \frac{\bar{f}_j^I \tilde{q}_j^I}{\tilde{\lambda}_0^I - \underline{f}_j^I} + h \frac{(1+G)(\tilde{\lambda}_0^I)^{-1} \varepsilon}{\tilde{\lambda}_0^I - (h - \varepsilon)} &> \sum_{j=1}^I \frac{\bar{f}_j^I \tilde{q}_j^I}{\tilde{\lambda}_0^I - \underline{f}_j^I} \\ &\geq \int_0^h \frac{xg(x)}{\lambda_0 - x} \\ &= 1, \end{aligned}$$

which proves the claim, by monotonicity.

Take any $\bar{\lambda}_0 > \lambda_0$. We show that eventually, $\tilde{\lambda}_0^I < \bar{\lambda}_0$. Let

$$\mathcal{I}(\lambda) = \int_0^h \frac{xg(x)}{\lambda - x} dx,$$

and note that $\mathcal{I}(\bar{\lambda}_0) < 1$. From (25), we get

$$\begin{aligned} \sum_{j=1}^I \frac{\bar{f}_j^I \tilde{q}_j^I}{\bar{\lambda}_0 - \underline{f}_j^I} &= \varepsilon \sum_{j=1}^I \frac{\tilde{q}_j^I}{\bar{\lambda}_0 - \underline{f}_j^I} + \sum_{j=1}^I \frac{\underline{f}_j^I \tilde{q}_j^I}{\bar{\lambda}_0 - \underline{f}_j^I} \\ &\leq \varepsilon \int_0^h \frac{g(x)}{\bar{\lambda}_0 - x} + \int_0^h \frac{xg(x)}{\bar{\lambda}_0 - x} \\ &\leq \varepsilon(1+G)(\lambda_0)^{-1} + \mathcal{I}(\bar{\lambda}_0). \end{aligned}$$

As for the other term in (20), note that as soon as

$$\bar{\lambda}_0 \geq \lambda_0(1 + (1+G)(\lambda_0)^{-1}\sqrt{\varepsilon}) \geq h(1 + (1+G)(\bar{\lambda}_0)^{-1}\sqrt{\varepsilon}) - \varepsilon,$$

(the second inequality is always true), we have

$$h \frac{(1+G)(\bar{\lambda}_0)^{-1}\varepsilon}{\bar{\lambda}_0 - (h - \varepsilon)} \leq \sqrt{\varepsilon}.$$

Therefore,

$$\sum_{j=1}^I \frac{\bar{f}_j^I \tilde{q}_j^I}{\bar{\lambda}_0 - \underline{f}_j^I} + h \frac{(1+G)(\bar{\lambda}_0)^{-1}\varepsilon}{\bar{\lambda}_0 - (h - \varepsilon)} \leq \sqrt{\varepsilon} + \varepsilon(1+G)(\lambda_0)^{-1} + \mathcal{I}(\bar{\lambda}_0) < 1,$$

for I large enough, which proves the claim by (20) and monotonicity. Furthermore, since $\bar{\lambda}_0 > \lambda_0$ is arbitrary, we have (26). This concludes the proof. ■

C.3 Innovation-Pays-Off Phase

Assume the density g is defined on $[0, h]$ with $h < +\infty$ and assume further that $g(x) > 0$ for all $x \in (0, h)$ (we allow 0 at the endpoints). In this section, we consider the case

$$\mathcal{I}(h) := \int_0^h \frac{xg(x)}{h-x} dx < 1. \quad (27)$$

We also assume

$$\int_0^h g(x) dx = 1, \quad (28)$$

although this is not necessary.

Example 4 Consider the case where \mathcal{Q} is Beta(α, β). Then it is easy to show that

$$\int_0^1 \frac{xg(x)}{1-x} dx = \frac{B(\alpha+1, \beta-1)}{B(\alpha, \beta)} = \frac{\alpha}{\beta-1},$$

where B is the Beta function. Therefore, (27) is satisfied if $\beta > \alpha + 1$. This example is a generalization of the example given in [4].

By (27), there is no solution $\lambda_0 \geq h$ to

$$\int_0^h \frac{xg(x)}{\lambda_0 - x} dx = 1.$$

Instead, for $0 \leq a \leq b \leq h$, let

$$\nu_{[a,b]} = h \int_a^b \frac{g(x)}{h-x} dx.$$

Note in particular that

$$\nu_{[0,h]} = \int_0^h (h-x) \frac{g(x)}{h-x} dx + \int_0^h \frac{xg(x)}{h-x} dx = 1 + \mathcal{I}(h) < 2.$$

Also, for $n \geq 0$, let $M_{n,[a,b]}$ be the number of edges with an endpoint of fitness in $[a, b]$ (counting twice edges with two endpoints of fitness in $[a, b]$). For ease of notation, we note $M_{n,x} := M_{n,[x,x]}$.

We prove the following.

Theorem 7 (Continuous Case: Innovation-Pays-Off Phase) Assume g is defined on $[0, h]$ with $h < +\infty$ and assume further that $g(x) > 0$ for all $x \in (0, h)$ and

$$\int_0^h \frac{xg(x)}{h-x} dx < 1.$$

Then it holds that for all $0 \leq a < b < h$,

$$\frac{M_{n,[a,b]}}{n} \rightarrow \nu_{[a,b]}, \quad (29)$$

almost surely as $n \rightarrow +\infty$. Moreover, for $0 \leq a \leq h$, we have

$$\frac{M_{n,[a,h]}}{n} \rightarrow 2 - \nu_{[0,a]}, \quad (30)$$

almost surely as $n \rightarrow +\infty$.

Proof: The convergence (30) follows trivially from (29). Also, from the proof of Theorem 6 it follows that

$$\liminf_{n \rightarrow +\infty} \frac{M_{n,[a,b]}}{n} \geq \nu_{[a,b]},$$

almost surely for all $0 \leq a < b \leq h$ (replace λ_0 with h in the proof).

To obtain an upper bound, we consider the modified chain with fitness distribution \mathcal{Q}_ε with

$$g_\varepsilon(x) = \begin{cases} g(x), & 0 \leq x \leq h - \varepsilon, \\ 0, & x > h - \varepsilon. \end{cases}$$

It is clear that we can couple this modified chain with the original one so that for all $0 \leq a \leq b \leq h - \varepsilon$

$$M_{n,[a,b]} \leq M_{n,[a,b]}^{(\varepsilon)}.$$

(Proceed similarly to the proof of Lemma 1.) Also, from Example 3, it follows that the modified chain is in the Fit-Get-Richer phase which allows to apply Theorem 6 (this is the reason we allowed $G < 1$ in the proof of Theorem 6). Therefore, for all $0 \leq a \leq b \leq h - \varepsilon$,

$$\limsup_{n \rightarrow +\infty} \frac{M_{n,[a,b]}}{n} \leq \lambda_0^{(\varepsilon)} \int_a^b \frac{g(x)}{\lambda_0^{(\varepsilon)} - x} dx,$$

where $\lambda_0^{(\varepsilon)} \geq h - \varepsilon$ is a solution to

$$\int_0^{h-\varepsilon} \frac{xg(x)}{\lambda_0^{(\varepsilon)} - x} dx = 1.$$

We claim that $\lambda_0^{(\varepsilon)} \rightarrow h$ as $\varepsilon \rightarrow 0$ which proves (29). Indeed, note that

$$\int_0^{h-\varepsilon} \frac{xg(x)}{h-x} dx \leq \int_0^h \frac{xg(x)}{h-x} dx < 1.$$

Therefore, $h - \varepsilon \leq \lambda_0^{(\varepsilon)} < h$. This concludes the proof. ■

C.4 Unbounded Case

The unbounded fitness case also follows easily from the previous proof (see also the proof in the discrete case). Therefore, we state the result without proof.

Theorem 8 (Continuous Case: Unbounded Case) *Assume g is defined on $[0, +\infty)$. Assume further that $g(x) > 0$ for all $x \in (0, +\infty)$ and*

$$\int_0^{+\infty} g(x) dx = 1.$$

Then it holds that for all $0 \leq a < b < +\infty$,

$$\frac{M_{n,[a,b]}}{n} \rightarrow \int_a^b g(x) dx,$$

almost surely as $n \rightarrow +\infty$. Moreover, for $0 \leq a < +\infty$, we have

$$\frac{M_{n,[a,+\infty)}}{n} \rightarrow 2 - \int_0^a g(x) dx,$$

almost surely as $n \rightarrow +\infty$.